the admittance matrix of Fig. 11(b) is $(y_{ij}{}^e+y_{ij}{}^o)/2$. The connection of one-to-one ideal transformers in the equivalent circuit of Fig. 10 (and Fig. 3) is necessary since the flow of loop currents between the partial networks $z_{ij}{}^{oe}$ or $y_{ij}{}^{oe}$ in Fig. 11 must be prevented.

To conclude our proof, the off-diagonal submatrices $(\mathbf{z}_{ij}{}^e-\mathbf{z}_{ij}{}^o)/2$ or $(\mathbf{y}_{ij}{}^e-\mathbf{y}_{ij}{}^o)/2$ still remain to be identified. The diagonal elements of $(\mathbf{z}_{ij}{}^e-\mathbf{z}_{ij}{}^o)/2$ are, by definition, the transfer impedance between port i and $i+N$, or j and $j+N$, in Fig. 10(b). Here we notice that the difference between the corresponding diagonal elements of $(\mathbf{z}_{ij}{}^e-\mathbf{z}_{ij}{}^o)/2$ and of $(\mathbf{z}_{ij}{}^e+\mathbf{z}_{ij}{}^o)/2$ is merely the sign between the partial networks $\mathbf{z}_{ij}{}^e$ and $\mathbf{z}_{ij}{}^o$. This is shown clearly in Fig. 10 where i and $i+N$ are the network ports of a two-port lattice network when j and $j+N$ are open-circuited and vice versa. Similarly, Fig. (12a) may be employed to show that the off-diagonal elements of $(\mathbf{z}_{ij}{}^e-\mathbf{z}_{ij}{}^o)/2$ in the equivalent circuit of Fig. 10(a)are the transfer impedances between the corresponding network ports of Fig. 10(b). Analogous considerations apply to the identification of $(\mathbf{y}_{ij}{}^{e}_{o}-\mathbf{y}_{ij}{}^{e}_{o})/2$ as the transfer admittances between the network ports of Fig. 10(a) [see Fig. 11(b)].

Since the mode indices i and j are chosen arbitrarily in the preceding discussion, the proof of generality for the $2N$-port lattice network is complete. Thus, we may conclude that the network given in Fig. 3 is capable of representing any lossless, symmetrical, $2N$-port structure characterized as in (2) or (4).

ACKNOWLEDGMENT

The author wishes to thank Dr. L. B. Felsen for his guidance and his many valuable suggestions.

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Perturbation Theorems for Waveguide Junctions, with Applications

D. M. KERNS AND W. T. GRANDY, JR.

***Abstract*—Perturbation theorems are derived in the context of a theory of waveguide junctions. These theorems express changes in impedance or admittance matrix elements, due to changes in a waveguide junction, in terms of integrals over products of perturbed and unperturbed basis fields associated with the junction and with its adjoint. Media involved are required only to be linear.**

Concepts of first-order perturbation theory are discussed briefly, and the term "correct to the lowest order" is precisely defined. The need of explicit theorems telling when one may expect results actually correct to the lowest order is noted.

Two problems are solved approximately by the perturbation approach:

1) reflection at the junction of rectangular waveguide with filleted waveguide of the same main dimensions; and

2) the effect of finite conductivity of both obstacle and waveguide wall for half-round inductive obstacles in rectangular waveguide.

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I. INTRODUCTION

THE PURPOSE of this paper is to present certain perturbation theorems in the context of a theory of waveguide junctions, to discuss briefly some of the rationale and the peculiarities of the simplest applications of perturbation methods, and to solve several problems that are illustrative as well as useful.

The presentation of the theorems in Section III of this paper was inspired largely by a paper by Monteath,[1] which gives theorems of the same type, but in a

[1] G. D. Monteath, "Application of the compensation theorem to certain propagation and radiation problems," *Proc. IEE* (*London*), pt. IV, vol. 98, pp. 23–30, 1951.

different context. The similarity in form is perhaps greater than that in content. Other related theorems have also appeared in the literature.[2,3] The theorems given here are sufficiently general to enable one, in principle, to consider arbitrary linear media.

The discussion, which relates mainly to the concept of approximations "correct to the lowest order" and to the use of unperturbed fields as approximations for perturbed ones, serves as a link between the general theorems and the applications made here.

The two problems considered are: 1) the junction of rectangular waveguide with filleted waveguide of the same main dimensions. This problem partly simulates the junction of precision and commercial waveguide, and the results are of interest in the development of precise reflection coefficient (or impedance) measurement techniques;[4] 2) the effect of finite conductivity of both obstacle and waveguide wall for half-round inductive obstacles in rectangular waveguide. The solution of this problem is intended to enhance the usefulness of the half-round obstacles as calculable standards of waveguide reflection coefficient (or impedance).

II. Electromagnetic Formulation

The required formulation of the elements of a theory of waveguide junctions has been given elsewhere.[5] We restate briefly the results needed for this paper, with certain simplifications and adaptations.

For our purposes, a waveguide junction is a linear electromagnetic system possessing ideal waveguide leads and is subject to excitation only through non-attenuated modes in these leads. The domain of the electromagnetic field is the (finite) region V with complete boundary S and inward normal $\boldsymbol{n}$ on S. The surface S consists of a part S_0, on which (in the unperturbed junction) tangential electric or magnetic fields vanish, and the parts $S_1, S_2, \cdots, S_n$, where S_m is the terminal surface in the mth of the n waveguide leads (Fig. 1). Within V, the complex vectors $\boldsymbol{E}$, $\boldsymbol{H}$ of the time-harmonic electromagnetic field satisfy Maxwell's equations, which are written

$$\boldsymbol{E} = \mathcal{E}(\boldsymbol{H}), \qquad \boldsymbol{H} = \mathcal{H}(\boldsymbol{E})$$

using the operators

$$\mathcal{E} \equiv (j\omega\epsilon)^{-1}\cdot\nabla\times \qquad \mathcal{H} \equiv -(j\omega\mu)^{-1}\cdot\nabla x \qquad (1)$$

as abbreviations. Here j is the imaginary unit, $\omega/(2\pi)$

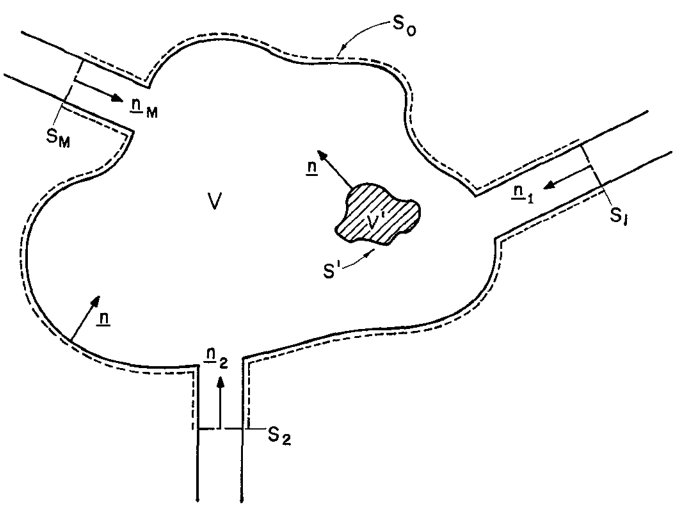

Fig. 1. Schematic illustration of regions V, V', and surfaces $S_0, S_1, \cdots, S_n$.

is the frequency, and μ, ϵ are, in general, complex nonsymmetric dyadic point-functions, which reduce to real scalar constants in the ideal portions of the waveguides.

The tangential components $\boldsymbol{E}_t$, $\boldsymbol{H}_t$, of $\boldsymbol{E}$, $\boldsymbol{H}$ on S_m are expressible in the form

$$\boldsymbol{E}_t = \sum_{\mu=1}^{\nu_m} v_{m\mu}\boldsymbol{e}_{m\mu}{}^0, \qquad \boldsymbol{H}_t = \sum_{\mu=1}^{\nu_m} i_{m\mu}\boldsymbol{h}_{m\mu}{}^0. \qquad (2)$$

Here ν_m is the number of propagated modes in the mth waveguide, $v_{m\mu}$ and $i_{m\mu}$ are scalar coefficients, and the terminal basis-fields $\boldsymbol{e}_{m\mu}{}^0$ and $\boldsymbol{h}_{m\mu}{}^0$ are real and subject to the power normalization

$$\int_{S_m} \boldsymbol{e}_{m\mu}{}^0\boldsymbol{h}_{m\lambda}{}^0\boldsymbol{n}_m\, dS = \delta_{\mu\lambda}, \qquad (3)$$

where $\delta_{\mu\lambda}$ is a Kronecker delta and $\boldsymbol{n}_m$ denotes $\boldsymbol{n}$ on S_m; here and subsequently, integrands in surface integrals are scalar triple products. Impedance normalization is given by the relation

$$\boldsymbol{h}_{m\mu}{}^0 = \zeta_{m\mu}{}^0\eta_{m\mu}\boldsymbol{n}_m \times \boldsymbol{e}_{m\mu}{}^0, \qquad (4)$$

where $\eta_{m\mu}$ is the wave-admittance of mode μ in waveguide m, and $\zeta_{m\mu}{}^0$ is the arbitrary characteristic impedance of this mode. These normalizations determine the terminal basis-fields up to the choice of a sign.

On S_0, the homogeneous boundary condition $\boldsymbol{n}\times\boldsymbol{E}=0$ applies. The additional prescription $v_{l\lambda}=\delta_{lm}\delta_{\lambda\mu}$ for given m and μ determines [through (2)] an electromagnetic field in V, which is denoted $\boldsymbol{e}_{m\mu}$, $\mathcal{H}(\boldsymbol{e}_{m\mu})$. Similarly, the prescription $i_{l\lambda}=\delta_{lm}\delta_{\lambda\mu}$ determines an electromagnetic field denoted $\mathcal{E}(\boldsymbol{h}_{m\mu})$, $\boldsymbol{h}_{m\mu}$. The fields $\boldsymbol{e}_{m\mu}$ and $\boldsymbol{h}_{m\mu}$ are appropriately called electric and magnetic junction basis-fields, respectively. If we now define the impedance matrix Z and the admittance matrix Y of the junction by writing

$$v_{l\lambda} = \sum_{m\mu} Z_{l\lambda,m\mu}i_{m\mu}, \qquad i_{l\lambda} = \sum_{m\mu} Y_{l\lambda,m\mu}v_{m\mu}, \qquad (5)$$

2 V. H. Rumsey, "The reaction concept in electromagnetic theory," *Phys. Rev.*, vol. 94, pp. 1483–1491, 1954. See also "Errata," *Phys. Rev.*, vol. 95, p. 1705, 1954.

3 A. G. Redfield, "An electrodynamic perturbation theorem, with applications to non-reciprocal systems," *J. Appl. Phys.*, vol. 25, pp. 1021–1024, 1954.

4 Experimental measurements pertaining to this problem were described by W. J. Anson and R. W. Beatty at the *1962 PGMTT Nat'l. Symp.*, Boulder, Colo.

5 D. M. Kerns, "Analysis of symmetrical waveguide junctions," *J. Res. NBS*, vol. 46, pp. 267–282, Apr. 1951.

then we have for the matrix elements the basic expressions

$$Z_{l\lambda,m\mu} = \int_{S_l} \mathcal{E}(\boldsymbol{h}_{m\mu})\boldsymbol{h}_{l\lambda}\boldsymbol{n}_l \, dS,$$

$$Y_{l\lambda,m\mu} = \int_{S_l} \boldsymbol{e}_{l\lambda}\mathcal{H}(\boldsymbol{e}_{m\mu})\boldsymbol{n}_l \, dS. \quad (6)$$

These may be verified with the aid of (1)–(3), (5), and the definitions of the junction basis-fields.

In addition to the unperturbed system having the parameters μ, ϵ and the boundary condition $\boldsymbol{n}\times\boldsymbol{E}=0$ holding on S_0, we consider also a changed or perturbed system in which the parameters μ', ϵ' or the boundary condition on S_0 (or both) may differ from the corresponding properties of the first system. For simplicity, we admit nothing more complicated than a scalar impedance boundary condition on S_0 in the perturbed junction, and we shall not consider perturbations that would change the terminal basis-fields.

We must consider also the systems "adjoint" to the original system and to the perturbed system, respectively. By the "adjoint" to a given system is here meant one having parameters $\hat{\mu}$, $\hat{\epsilon}$ equal, respectively, to the transposes of the μ, ϵ of the given system and having exactly the same boundary conditions as the given system. (This slightly restricted definition is adequate for the present situation.)

The region V and the terminal basis-fields are the same for all four systems involved. Quantities associated with a perturbed system or with an adjoint system are distinguished throughout by primes or circumflexes, respectively.

It seems that, in general, there is no simple relation between the field in a given junction and the field in its adjoint. Knowledge of such relations would, of course, be important in applications. Special cases where such relations do exist are shown below. The statements may be verified with the aid of Maxwell's equations, with due regard for boundary conditions permitted or prescribed. If μ, ϵ are symmetric (reciprocity condition), one finds

$$\hat{\boldsymbol{e}}_q = \boldsymbol{e}_q,\ \hat{\mathcal{H}}(\hat{\boldsymbol{e}}_q) = \mathcal{H}(\boldsymbol{e}_q);\quad \hat{\mathcal{E}}(\hat{\boldsymbol{h}}_q) = \mathcal{E}(\boldsymbol{h}_q),\ \hat{\boldsymbol{h}}_q = \boldsymbol{h}_q; \quad (7)$$

and the term "self-adjoint" is appropriate. If μ, ϵ are Hermitian and $\boldsymbol{n}\times\boldsymbol{E}=0$ on S_0 (absence of dissipation), then

$$\hat{\boldsymbol{e}}_q = \bar{\boldsymbol{e}}_q,\ \hat{\mathcal{H}}(\hat{\boldsymbol{e}}_q) = -\overline{\mathcal{H}(\boldsymbol{e}_q)};$$

$$\hat{\mathcal{E}}(\hat{\boldsymbol{h}}_q) = -\overline{\mathcal{E}(\boldsymbol{h}_q)},\ \hat{\boldsymbol{h}}_q = \bar{\boldsymbol{h}}_q; \quad (8)$$

the superposed bar denotes the complex conjugate. If μ, ϵ are symmetric and Hermitian (=real symmetric), then both (7) and (8) hold. The junction basis-fields are then restricted to be pure real, and the associated fields pure imaginary.

We now observe that the immittance expressions (6) may be rewritten as follows:

$$Z_{l\lambda,m\mu} = \int_\Sigma \mathcal{E}(\boldsymbol{h}_{m\mu})\hat{\boldsymbol{h}}_{l\lambda}'\boldsymbol{n}\, dS,$$

$$Y_{l\lambda,m\mu} = \int_\Sigma \hat{\boldsymbol{e}}_{l\lambda}'\mathcal{H}(\boldsymbol{e}_{m\mu})\boldsymbol{n}\, dS, \quad (9)$$

where $\Sigma = S_1+S_2+\cdots+S_n$. The extension of the integrals to go over all the terminal surfaces is purely formal at this stage, since the tangential components of the basis-fields involved vanish on all but the lth terminal surface. The use of the basis-fields $\hat{\boldsymbol{h}}_{l\lambda}'$ and $\hat{\boldsymbol{e}}_{l\lambda}'$ (instead of $\boldsymbol{h}_{l\lambda}$ and $\boldsymbol{e}_{l\lambda}$, which respectively have identical tangential components on the terminal surfaces) indicates the continuation into V to be taken in considering volume-integral expressions in Section III. For the adjoint of the changed junction, the expressions corresponding to (9) are

$$\hat{Z}'_{m\mu,l\lambda} = \int_\Sigma \hat{\mathcal{E}}'(\hat{\boldsymbol{h}}_{l\lambda}')\boldsymbol{h}_{m\mu}\boldsymbol{n}\, dS,$$

$$\hat{Y}'_{m\mu,l\lambda} = \int_\Sigma \boldsymbol{e}_{m\mu}\hat{\mathcal{H}}'(\hat{\boldsymbol{e}}_{l\lambda}')\boldsymbol{n}\, dS. \quad (10)$$

Here the basis-fields $\boldsymbol{e}_{m\mu}$ and $\boldsymbol{h}_{m\mu}$ are used advisedly.

III. Perturbation Theorems

In what follows it will suffice to use single-letter indices p, q, $\cdots$, to indicate both waveguide and mode.[6]

The immittance elements of a given system and its adjoint satisfy the "adjoint reciprocity" relation;[7] e.g., stated for the changed system,

$$Z_{pq}' = \hat{Z}_{qp}', \qquad Y_{pq}' = \hat{Y}_{qp}'. \quad (11)$$

Using the first of these equations and the first equations in (9) and (10), one may find

$$Z_{pq}' - Z_{pq} = \int_\Sigma [\hat{\mathcal{E}}'(\hat{\boldsymbol{h}}_p')\boldsymbol{h}_q - \mathcal{E}(\boldsymbol{h}_q)\hat{\boldsymbol{h}}_p']\boldsymbol{n}\, dS. \quad (12)$$

Let V', bounded by S', denote the subregion of V in which one or both of the constitutive parameters in the changed system actually differ from those in the unchanged system (see Fig. 1). It is easily shown that the expression in brackets in the integrand of (12), considered as a (vector) function of position in V, has zero divergence in the region $V-V'$. The desired theorems for changes of impedance follow from this property with the aid of the divergence theorem.

If perturbations occur in the boundary conditions on

[6] It should be pointed out that the theorems that follow could be stated as well for sets of basis-fields other than the particular ones defined previously. "Change of basis" is discussed in D. M. Kerns.[5]

[7] Equations (11) are immediate consequences of theorems given by M. H. Cohen, "Reciprocity theorem for anisotropic media," *Proc. IRE* (*Correspondence*), vol. 43, p. 103, January 1955. For a derivation wholly in the waveguide-junction context, see *Electromagnetic Theory and Antennas*. Proceedings of a 1962 Symposium held at Copenhagen, Denmark, E. C. Jordan, Ed., New York: Pergamon 1963, p. 253 ff.

S_0 but not in the parameters within V, then

$$Z_{pq}' - Z_{pq} = -\int_{S_0} [\hat{\mathcal{E}}'(\hat{\boldsymbol{h}}_p')\boldsymbol{h}_q - \mathcal{E}(\boldsymbol{h}_q)\hat{\boldsymbol{h}}_p']\boldsymbol{n}\, dS. \quad (13)$$

If perturbations are made in the parameters within V but not in the boundary conditions on S_0, then

$$Z_{pq}' - Z_{pq} = -\int_{S'} [\hat{\mathcal{E}}'(\hat{\boldsymbol{h}}_p')\boldsymbol{h}_q - \mathcal{E}(\boldsymbol{h}_q)\hat{\boldsymbol{h}}_p']\boldsymbol{n}\, dS, \quad (14)$$

where $\boldsymbol{n}$ on S' is directed into $V-V'$. In problems that involve only finite perturbations of the constitutive parameters, a volume-integral form of (14) may be useful. Again using the divergence theorem, one obtains

$$Z_{pq}' - Z_{pq} = j\omega \int_{V'} [\hat{\boldsymbol{h}}_p' \cdot (\mu' - \mu) \cdot \boldsymbol{h}_q - \hat{\mathcal{E}}'(\hat{\boldsymbol{h}}_p') \cdot (\epsilon' - \epsilon) \cdot \mathcal{E}(\boldsymbol{h}_q)]\, dV. \quad (15)$$

The integrand clearly vanishes outside of the subregion V'.

The expressions for changes in the admittance-matrix elements corresponding to (13), (14), and (15) are as follows. For perturbations of the boundary con-tion on S_0,

$$Y_{pq}' - Y_{pq} = -\int_{S_0} [\boldsymbol{e}_q\hat{\mathcal{H}}'(\hat{\boldsymbol{e}}_p') - \hat{\boldsymbol{e}}_p'\mathcal{H}(\boldsymbol{e}_q)]\boldsymbol{n}\, dS; \quad (16)$$

for perturbations of the constitutive parameters within V,

$$Y_{pq}' - Y_{pq} = -\int_{S'} [\boldsymbol{e}_q\hat{\mathcal{H}}'(\hat{\boldsymbol{e}}_p') - \hat{\boldsymbol{e}}_p'\mathcal{H}(\boldsymbol{e}_q)]\boldsymbol{n}\, dS; \quad (17)$$

and for finite perturbations of the parameters in V,

$$Y_{pq}' - Y_{pq} = j\omega \int_{V'} [\hat{\boldsymbol{e}}'_p \cdot (\epsilon' - \epsilon) \cdot \boldsymbol{e}_q - \hat{\mathcal{H}}'(\hat{\boldsymbol{e}}_p') \cdot (\mu' - \mu) \cdot \mathcal{H}(\boldsymbol{e}_q)]\, dV. \quad (18)$$

Expressions for changes in immittance when boundary and volume perturbations are simultaneously involved consist of sums of the previous appropriate expressions.

IV. Discussion

The foregoing equations give exact changes in the immittance-matrix elements, but require that certain fields in both the changed and the unchanged junction be known. (They are not integral equations and, therefore, do not in themselves provide a means of determining the needed fields.) In the type of problems to be discussed here, the original system is "simple" (i.e., basis-fields may be obtained practicably), and the perturbations may be considered in some sense small. This, of course, suggests approximating the needed perturbed fields by unperturbed ones. This expedient is connoted by the term "first-order perturbation theory," is frequently used in both electromagnetic and quantum-mechanical eigenvalue problems, and is adopted here. In the present context, as well as in the application to eigenvalue problems, one hopes to obtain at least a result "correct to the lowest order." This concept is important for the present discussion. Its definition is based upon the hope that the true result, e.g., an impedance change, is expressible in a power series[8]

$$Z' - Z = c_1p + c_2p^2 + \cdots, \quad (19)$$

where p is a suitable parameter representing the scale or magnitude of the perturbation [e.g., if ϵ is changed homogeneously in a region, one could have $p=\epsilon'-\epsilon$; if the change is not homogeneous, one may introduce p artificially as a multiplier as in the expression $p(\epsilon'-\epsilon)$.] To obtain the "correct lowest-order result" means to obtain correctly the first nonvanishing term on the right side of (19).

Because of the obvious attractiveness of obtaining higher-order results with zero-order input and without the necessity of constructing a Green's function, it would be well if one had theorems telling when one could expect a result actually correct to the lowest order. No such theorems in a form immediately applicable to the problems of the type considered here appear to have been published (nor are such theorems given here). That such theorems are needed is certainly indicated by examples which show that the procedure sometimes does and sometimes does not give correct lowest-order results. (This is discussed in somewhat more detail in the following.) Furthermore, convergence theorems furnishing estimates of radii of convergence would also be helpful in estimating rate of convergence and, hence, the adequacy of a given lowest-order approximation.

Of course, one might expect even an incorrect lowest-order result to be dimensionally correct and, thus, qualitatively of some usefulness—as in curve fitting. However, such a result is certainly likely to be of a lower order of usefulness than a correct lowest-order result!

Problems in which ϵ or μ, or both, are changed in the subregion V' can be regarded as forming a more or less distinct class. Problems of this class appear relatively tractable from the point of view of finding convergence criteria. The underlying mathematical problem can be put in integral equation form, and the series (19) corresponds to the Liouville-Neumann series solution of the integral equation. The typical behavior of this form of solution[9] indicates that for sufficiently small changes in ϵ and μ occurring in a sufficiently small region, (19) should converge (and hence, in particular, a first-order perturbation approach should yield a correct lowest-order result), but that otherwise (19) should

[8] Concerning the quantum-mechanical eigenvalue problem, K. O. Friedrichs remarks in an unpublished report, "This hope will be fulfilled only under favorable circumstances."

[9] See, e.g., Courant-Hilbert, *Methods of Mathematical Physics*. 1st English ed., New York: Interscience, 1953, p. 140 ff; or F. Riesz-B. Sz.-Nagy, *Functional Analysis*. New York: Ungar, 1955, p. 143 ff.

diverge. Preliminary convergence criteria of this character have, in fact, been obtained for certain immittance perturbation problems in rectangular waveguide.

The first problem considered in the next section may be regarded as a member of a class of problems in which a perfectly conducting boundary is indented or a perfectly conducting object is introduced into the region V of the waveguide junction. If these changes are regarded as a change of conductivity from 0 to infinity in the subregion V', then the use of the volume-integral formulas (15), (18) would require the application of limiting processes that might well be difficult in general. Nevertheless, the surface-integral formulas (14), (17) remain directly applicable, and it is interesting to attempt to apply the concepts of first-order perturbation theory to them.

To simplify the discussion, we consider only diagonal elements of the immittance matrices of a self-adjoint, nondissipative junction. This has the advantage that the resulting formulas are closely related to certain well-known formulas for the perturbation of the resonance frequencies of electromagnetic cavities, and enables us to benefit from some results and theory obtained in connection with the cavity problems. In (14) and (17), we note that the tangential components of the perturbed electric fields vanish on the surface of the obstacle, replace the perturbed magnetic fields by the unperturbed ones, apply the divergence theorem, and, thus, obtain

$$Z_{qq}' - Z_{qq} = j\omega \int_{V'} \{\epsilon \,|\, [\mathcal{E}(\boldsymbol{h}_q)]^2 \,|\, - \mu \boldsymbol{h}_q^2\} \, dV, \quad (20a)$$

$$Y_{qq}' - Y_{qq} = j\omega \int_{V'} \{\mu \,|\, [\mathcal{H}(\boldsymbol{e}_q)]^2 \,|\, - \epsilon \boldsymbol{e}_q^2\} \, dV. \quad (20b)$$

In writing these equations, we have used (7) and (8) and we have made the algebraic signs as explicit as possible. The analogous perturbation formula for frequency shifts is[10]

$$\omega' - \omega = \frac{\omega}{2W} \int_{V'} (\epsilon \boldsymbol{E} \cdot \overline{\boldsymbol{E}} - \mu \boldsymbol{H} \cdot \overline{\boldsymbol{H}}) \, dV, \qquad (21)$$

where

$$W = \int_V \mu \boldsymbol{H} \cdot \overline{\boldsymbol{H}} \, dV = \int_V \epsilon \boldsymbol{E} \cdot \overline{\boldsymbol{E}} \, dV.$$

A convenient and cogent illustration of the possible behavior of formulas of the type of (20) and (21) is afforded by some results obtained by Bolle,[11] who considered the perturbation of certain eigenfrequencies of a circularly cylindrical cavity produced by a concentric, symmetrically located circularly cylindrical disk. He computed several eigenfrequencies precisely as a function of the ratio of the thickness of the disk to the length of the cavity, and compared the precise results with results given by (21). The qualitative character of the results is shown clearly in Fig. 2: the result given by (21) is not correct to the lowest order—its graph is not tangent to the graph of the precise result even for a vanishingly small perturbation.

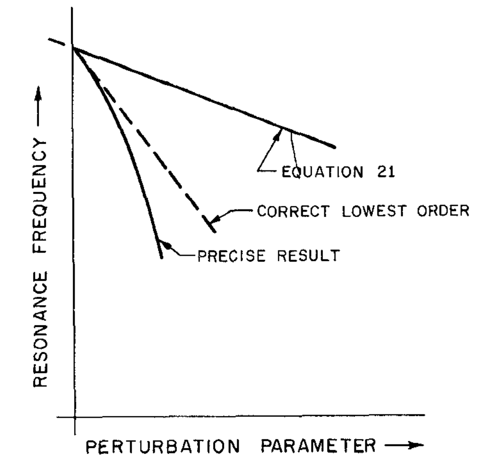

Fig. 2. Qualitative character of results obtained by Bolle.

By means of examples, for which results obtained by other methods are available, we have found that (20a) and (20b) yield correct lowest-order results in some cases and not in others. Two cases in which (20) yielded correct lowest-order results were the E-plane and H-plane steps in rectangular waveguide. These cases are mentioned because the first problem considered in the following section is somewhat similar: it is a plane discontinuity in rectangular waveguide involving small changes in both the broad and the narrow dimensions of the waveguide. Although this may be indicative, the best evidence that the result is correct to the order given is the good agreement of the computed results with experimental results (see Fig. 4).

In the second problem considered in the next section, the perturbation can be approximated as a change in surface impedance from zero to a small, finite value. Essentially this first-order perturbation approach has been used successfully many times in this type of problem, and there is no reason to doubt that the result is indeed correct to the lowest order. However, in this case, higher-order terms, as in (19), would be meaningless because of the approximation inherent in the use of the impedance boundary condition.

In both examples to be presented, it is apparent that the actual fields differ very little from the unperturbed fields over the major part of the integration domain in the perturbation formulas. Since the dominant contribution to the immittance changes comes from this

[10] J. Müller, "Untersuchungen über electromagnetische Hohlräume," *Z. Hochfrequenztechnik u. Elektroakustik*, vol. 54, pp. 157–161, 1939. R. Müller in ch. 2 of G. Goubau, *Electromagnetic Waveguides and Cavities*. New York: Pergamon, 1961, gives a more critical discussion and improved formulas that should benefit the immittance problem as well as the eigenvalue problem.

[11] D. M. Bolle, "Eigenvalues for a centrally loaded circular cylindrical cavity," *IRE Trans. on Microwave Theory and Techniques*, vol. MTT-10, pp. 133–138, March 1962.

region, one should perhaps expect good results. Unfortunately, while such an observation is certainly of pragmatic value, it is not sufficiently quantitative.

V. Applications

The two problems considered in this section involve only homogeneous isotropic media, and terminal surfaces are located in ideal rectangular waveguide. It is assumed that only the TE_{10} mode propagates in the ideal rectangular guide, and the waveguide characteristic impedance ζ^0 is set equal to the wave impedance $\omega\mu/\beta$, where β is the phase constant for the TE_{10} mode at the frequency of operation. Terminal basis-fields, satisfying (3) and (4) with the forementioned choice of characteristic impedance, and appropriate for a terminal surface located at $z = -L \leq 0$, are

$$\boldsymbol{e}^0 = \sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a} \boldsymbol{e}_y, \qquad \boldsymbol{h}^0 = -\sqrt{\frac{2}{ab}} \sin \frac{\pi x}{a} \boldsymbol{e}_x, \tag{22}$$

where $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are unit vectors of the coordinate system $0xyz$ shown in Fig. 3, and a and b are, respectively, the broad and the narrow dimensions of the waveguide.

Junction of Rectangular and Filleted Waveguide

The geometry and the geometrical parameters of this problem are shown in Fig. 3. We consider the fillets as regions in which the conductivity has been changed from zero to infinity, and seek to evaluate the input impedance Z_i at the reference plane $z = 0^-$, under the assumption that the perturbed waveguide extends to $z = \infty$. Equation (14) is applicable and yields

$$Z_i' - \zeta^0 = \int_{S'} \mathcal{E}(\boldsymbol{h})\boldsymbol{h}'\boldsymbol{n}\, dS. \tag{23}$$

The unperturbed electromagnetic field is that of a suitably normalized TE_{10} mode of the rectangular guide, traveling in the $+z$ direction. In fact,

$$\mathcal{E}(\boldsymbol{h}) = \zeta^0 \boldsymbol{e}^0 e^{-j\beta z},$$

$$\boldsymbol{h} = \left(\boldsymbol{h}^0 + \frac{j\pi}{\beta a}\sqrt{\frac{2}{ab}} \cos \frac{\pi x}{a} \boldsymbol{e}_z\right) e^{-j\beta z}.$$

(One may note that the normalization is chosen so that the transverse component of $\boldsymbol{h}$ at the terminal surface is equal to $\boldsymbol{h}^0$, in accordance with the definition of magnetic junction basis-fields in Section II.) We replace $\boldsymbol{h}'$ by $\boldsymbol{h}$ in (23) and evaluate to the lowest order in R the radius of the fillets. The contribution of the integration over the face of the discontinuity at $z = 0$ is found to be $O(R^4)$.[12] The significant contributions, which involve an integration from $z = 0$ to ∞, are easily evaluated using the familiar artifice of assuming an infinitesimal attenuation to secure convergence. The result is

$$Z_i' = \zeta^0 + \zeta^0 \frac{R^2\pi^2}{\beta^2 b a^3}(4 - \pi) + O(R^4).$$

The reflection coefficient at the junction of the two waveguides is then

$$S_{11} = \frac{Z_i' - \zeta^0}{Z_i' + \zeta^0} = \left(\frac{\lambda_g}{a}\right)^2 \frac{R^2}{ab}\left(\frac{4 - \pi}{8}\right) + O(R^4), \tag{24}$$

where λ_g is the (unperturbed) guide wavelength.

The good agreement of calculated results based on the foregoing equation with the results of careful measurements made by Anson and Beatty[4] is shown in Fig. 4.

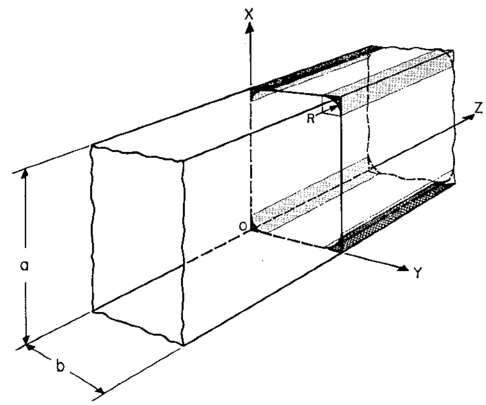

Fig. 3. Junction of rectangular and filleted waveguides.

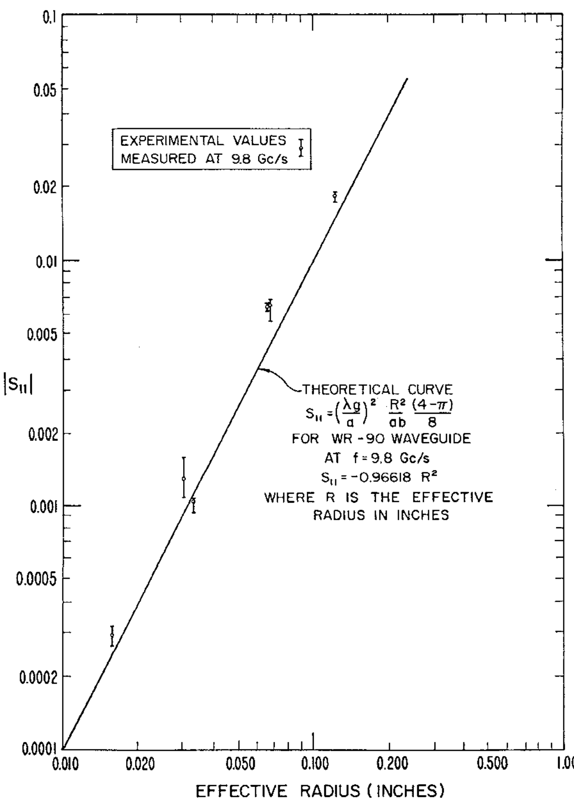

Fig. 4. Comparison of calculated and measured values of reflection coefficient for the junction of rectangular and filleted waveguides.

[12] The $O(x^n)$ notation should be understood as follows: $f(x) = O(x^n)$ implies that $|f(x)/x^n|$ remains bounded by a constant independent of x as $x \to x_0$. In this paper, $x_0 = 0$.

Finitely Conducting Half-Round Obstacle in Finitely Conducting Rectangular Waveguide

The unperturbed junction is the perfectly conducting half-round obstacle in perfectly conducting waveguide shown in Fig. 5; the perturbation is the change of conductivity of both obstacle and waveguide from infinite to finite but large values (not necessarily the same for obstacle and for waveguide). We wish to evaluate the input admittance (and the reflection coefficient) of the perturbed junction at a terminal surface at $z=-L\leq 0$, under the assumption that the waveguide on the right extends to $z=\infty$. This assumption is equivalent to the assumption of a reflectionless termination on the right and corresponds to a condition of practical application of the obstacles.

A network representation for the unperturbed problem is shown in Fig. 6. The impedance values of the elements entering into the representation of the obstacle are, each to the lowest order in the radius R of the obstacle,[13]

$$Z_{ee} = \frac{j\omega\mu a}{\pi}\left(\frac{a}{\pi R}\right)^2, \qquad Z_{oo} = -\frac{j\omega\mu a}{2\pi}\left(\frac{\pi R}{a}\right)^4. \qquad (25)$$

These values are given relative to the characteristic impedance $\zeta^0=\omega\mu/\beta$ and are referred to a terminal surface at $z=0$. We are assuming $R\ll a$, and in what follows, we shall consistently and without further comment retain only the lowest significant powers of R. By referring to Fig. 6 and employing the transmission-line equations, one may obtain for the input admittance of the structure

$$Y_1 = \eta^0 + 2Y_{ee}e^{-2j\beta L}. \qquad (26)$$

Here, for convenience, we have introduced the admittances $\eta^0=1/\zeta^0$ and $Y_{ee}=1/Z_{ee}$.

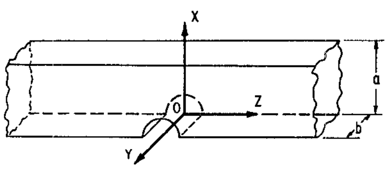

Fig. 5. Half-round inductive obstacle.

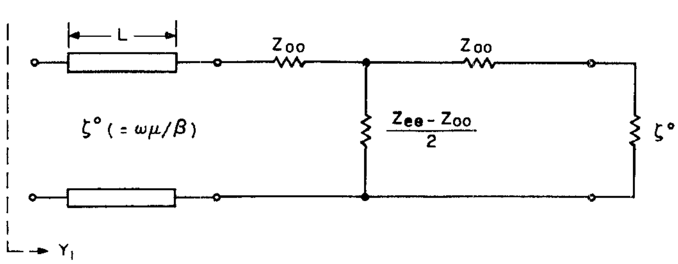

Fig. 6. Network representation of the unperturbed half-round obstacle problem.

[13] D. M. Kerns, "Half-round inductive obstacles in rectangular waveguide," *J. Res. NBS*, vol. 64B, pp. 113–130, 1960.

To calculate the change in Y_1, we need (16); this equation in this problem immediately reduces to

$$\Delta Y_1 = \int_{S_0} e'\mathcal{H}(e)n\, dS, \qquad (27)$$

where ΔY_1 denotes the change in Y_1. The perturbed electric field needed in this equation is approximated in the usual way by using the unperturbed magnetic field in the impedance boundary conditions, which is written

$$n \times e' \cong -Z_m h$$

(remembering that n is the inward normal on S_0). Here $Z_m=\omega\mu_m\delta_m(1+j)/2$; μ_m is the permeability and δ_m the skin depth in metal. The subscript m will be replaced by w or h, respectively, to denote the values associated with the waveguide or with the obstacle. Equation (27) thus becomes

$$\Delta Y_1 = Z_h \int_H [\mathcal{H}(e)]^2\, dS + Z_w \int_W [\mathcal{H}(e)]^2\, dS. \qquad (28)$$

The H integration goes over the hemicylindrical surface of the half-round; the W integration goes over the entire interior surface of the waveguide ($z\geq -L$), except for parts excluded by the obstacle. (For the surface-impedance approximation to be valid on the half round, the inequality $\delta_h \ll R$ must hold.)

The unperturbed electric field for the present problem may be obtained from previously published work[13] (but it is not given explicitly there). To the desired approximation,

$$e = C\left(\sin\frac{\pi x}{a} e^{-j\beta z} - \frac{\pi R^2}{ar}\sin\theta\right)e_y, \qquad (29)$$

where $C=\exp(-j\beta L)\sqrt{2/(ab)}$, and r, θ are polar coordinates such that $x=r\sin\theta$, $z=r\cos\theta$. The magnetic field associated with (29) is

$$\mathcal{H}(e) = \frac{jC}{\omega\mu}\left[\left(j\beta\sin\frac{\pi x}{a}e_x + \frac{\pi}{a}\cos\frac{\pi x}{a}e_z\right)e^{-j\beta z} - \frac{\pi R^2}{a}\nabla\times\left(\frac{\sin\theta}{r}e_y\right)\right]. \qquad (30)$$

It is interesting, and worth noting as a check, that the term involving R in this expression can be obtained as a result of another perturbation problem: If a circular cylinder (of radius R) with its axis parallel to e_y is placed in a homogeneous static (or quasi-stationary) magnetic field H_0e_z, the resulting perturbation of the field outside the cylinder is of the known form

$$-\frac{\mu-\mu_c}{\mu+\mu_c}H_0R^2\nabla\times\left(\frac{\sin\theta}{r}e_y\right), \qquad (31)$$

where μ_c is the permeability of the cylinder. If, further, H_0 is set equal to the z component of (30) evaluated at $x=0$, $z=0$ (omitting the term involving R), and if μ_c

is set equal to zero (to simulate a perfectly conducting cylinder), then (31) yields precisely the third term in (30).

Upon substituting (30) into (28) and carrying out the somewhat tedious integrations, one arrives at a result that can be conveniently written

$$\Delta Y_1 = \Delta\eta^0 + 2\Delta Y_{ee}e^{-2j\beta L}, \tag{32}$$

where $\Delta\eta^0$ and ΔY_{ee} have the following values

$$\Delta\eta^0 = \eta^0 \frac{(j-1)\mu_w\delta_w}{\mu}\left[\frac{1}{2b} - \left(\frac{\pi}{a\beta}\right)^2\left(\frac{1}{a} + \frac{1}{2b}\right)\right], \tag{33a}$$

$$\Delta Y_{ee} = Y_{ee}\frac{j-1}{\mu R}\left(\mu_h\delta_h - \frac{8}{3\pi}\mu_w\delta_w\right), \tag{33b}$$

and respectively denote the perturbations of η^0 and Y_{ee}.

The perturbation of the characteristic admittance is identified in the complete expression given by (28) simply as the value of ΔY_1 in the absence of an obstacle ($R=0$). The characteristic admittance (or impedance) of the TE_{10} mode in rectangular waveguide with lossy walls is thus obtained as a by-product in the foregoing calculation. This result is certainly not well-known and may be new. It is needed in the final calculations.

It should be observed that for the second term on the right in (32) to be a good approximation, it is necessary that $L \ll 1/\alpha$, where α is the attenuation constant for the TE_{10} mode in the lossy guide. This restriction will be removed in the final results.

It is noteworthy that ΔY_1 can be very small and may indeed vanish for reasonable values of the parameters involved. Thus, for example, if $a=2b$, then $\Delta\eta^0=0$ for $\omega/\omega_c=\pi/\sqrt{3}$, where ω_c is the cutoff angular frequency. The quantity ΔY_{ee} will vanish for suitable values of the ratio $\mu_h\delta_h/(\mu_w\delta_w)$; in fact ΔY_{ee} would be substantially zero for a copper half-round in a gold waveguide. The difference in sign between the two terms in parentheses in (33b) may be attributed to the fact that the obstacle tends to reduce the magnetic field on the waveguide walls in the immediate vicinity of the obstacle.

Probably the result of most immediate practical usefulness to be derived from the foregoing is the magnitude of the reflection coefficient corresponding to Y_1', as observed in lossy guide having the same properties as (or being a continuation of) the waveguide to the right of the terminal surface. For the magnitude of the reflection coefficient of the lossless obstacle in lossless waveguide, we have

$$|S_1| = \left|\frac{\eta^0 - Y_1}{\eta^0 + Y_1}\right| \cong \left|\frac{Y_{ee}}{\eta^0}\right|. \tag{34}$$

For the magnitude of the reflection coefficient of the lossy obstacle observed in lossy waveguide, we have

$$|S_1'| = \left|\frac{(\eta^0)' - Y_1'}{(\eta^0)' + Y_1'}\right| \cong \left|\frac{Y_{ee}}{\eta^0}\right|\left|1 - \frac{\Delta\eta^0}{\eta^0} + \frac{\Delta Y_{ee}}{Y_{ee}}\right| e^{-2\alpha L}.$$

Here, we have inserted the exponential factor exp $(-2\alpha L)$ in order to remove the restriction $L \ll 1/\alpha$. Expressions for the attenuation constant in rectangular waveguide operated above cutoff are well known and we consider α to be known. Finally, using (33) and (34), we obtain for the correction to the relative magnitudes of S_1 and S_1',

$$\left|\frac{S_1'}{S_1}\right| = \left\{1 + \frac{\mu_w\delta_w}{\mu}\left[\frac{1}{2b} - \left(\frac{\pi}{\beta a}\right)^2\left(\frac{1}{a} + \frac{1}{2b}\right)\right] + \frac{8\mu_w\delta_w}{3\pi\mu R} - \frac{\mu_h\delta_h}{\mu R}\right\} e^{-2\alpha L}. \tag{35}$$

It is fortunate, from the practical point of view, that this correction tends to be very small. This is especially so because effective values of surface impedance on both machined and electrodeposited surfaces are somewhat uncertain. The formula should serve to estimate the correction to reflection coefficients more accurate than those used in the derivation. The final formula is applicable as it stands to double half-round obstacles as well as to the single half-round obstacles. (The former consist of two opposed hemicylindrical indentations extending across the narrow sides of the guide.)